Introductory Exercises in Probability Calculus

1 Joint, marginal and conditionals I

The joint discrete probability table of $P_{A,B}(a, b)$ is given below:

		B		
		1	**2**	**3**
A	**1**	0.2	0.1	0.3
	2	0.1	0.1	0.2

Derive the following probabilities:

1. $P_{A,B}(1, 2)$
2. $P_B(2)$
3. $P_{A|B}(1|2)$

Are A and B independent?

2 Joint, marginal and conditionals II

Assume the probability densities $p(E \mid B)$, $p(B)$, $p(A, D \mid E)$, and $p(C \mid B, E)$ are known.

1. Draw the corresponding directed acyclic graph of the conditional probabilities to visualize the independence structure.
2. Derive $p(B, C, E)$
3. Derive the joint distribution of A, B, C, D, and E.
4. Derive $p(A, B \mid C, D, E)$

5. Derive $p(A \mid D)$
6. Derive $p(A \mid B, E)$

3 Compound distribution

Assume that:

$$\mu \sim f_\mu(m) = \begin{cases} 0.1\exp(-0.1m) & m \geq 0 \\ 0 & \text{else} \end{cases}$$

$$X \sim f_{X|\mu=m}(x \mid m) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x-m)^2}{2}\right)$$

That means X is normally distributed with mean μ and μ itself is exponentially distributed.

Derive and interpret:

1. $f_{X,\mu}(x, m)$
2. $f_X(x)$, a so called compound distribution.
3. $P(\mu > 5)$
4. $f_{X,\mu|\mu>5}(x, m)$
5. $f_{X|\mu>5}(x)$

It is not the aim to find closed forms for the integrals.

Introductory Exercises in Probability Calculus — solutions

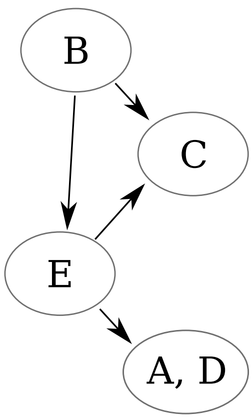
1 Joint, marginal and conditionals I

1. The joint probability $P_{A,B}(1,2) = 0.1$ can be read directly from the table.
2. The marginal $P_B(2)$ is computed by summing over all joint probabilities with $B = 2$, i.e. $P_{A,B}(1,2) + P_{A,B}(2,2) = 0.2$.
3. Using the results from above we can compute the conditional probability $P_{A|B}(1|2) = \frac{P_{A,B}(1,2)}{P_B(2)} = 0.5$.

A and B are not independent. To see this calculate first the marginal probabilities. In this example the joint probability is not equal to the product of the marginal probabilities, $P_{A,B}(a,b) \neq P_A(a)\,P_B(b)$, hence A and B cannot be independent.

2 Joint, marginal and conditionals II

1. DAG representation:



2. $p(B,C,E) = p(C \mid B,E)\, p(E \mid B)\, p(B)$

3. The joint distribution is

$$p(A,B,C,D,E) = p(C \mid B,E)\, p(E \mid B)\, p(B)\, p(A,D \mid E)$$

4.
$$p(A,B \mid C,D,E) = \frac{p(A,B,C,D,E)}{p(C,D,E)} = \frac{p(A,B,C,D,E)}{\int p(A,B,C,D,E)\, dA\, dB}$$

5.
$$\begin{aligned} p(A \mid D) = \frac{p(A,D)}{p(D)} &= \frac{\int p(A,B,C,D,E)\, dB\, dC\, dE}{\int p(A,B,C,D,E)\, dA\, dB\, dC\, dE} \\ &= \frac{\int p(A,D|E)\, p(E|B)\, p(B) dB\, dE}{\int p(A,D|E)\, p(E|B)\, p(B) dA\, dB\, dE} \end{aligned}$$

6.
$$\begin{aligned} p(A \mid B,E) &= \frac{\int p(A,D|E)\, p(E|B)\, p(B)\, dD}{p(E|B)\, p(B)} \\ &= \int p(A,D \mid E)\, dD = p(A \mid E) \end{aligned}$$

3 Compound distribution

1. joint distribution:

$$f_{X,\mu}(x,m) = f_{X|\mu}(x|m)\, f_\mu(m) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2}} 0.1 e^{-0.1m}$$

2. compound distribution

$$f_X(x) = \int_0^\infty f_{X,\mu}(x,m)\, dm$$

3. probability that μ is larger than 5:

$$P(\mu > 5) = \int_5^\infty f_\mu(m)\, dm = \exp(-0.1 \cdot 5)$$

4. joint distribution under the constraint that $\mu > 5$:

$$f_{X,\mu|\mu>5}(x,m) = \begin{cases} f_{X,\mu}(x,m)/P(\mu > 5) & \text{if } m > 5 \\ 0 & \text{otherwise} \end{cases}$$

5. marginal of X under constrain that $\mu > 5$:

$$f_{X|\mu>5}(x) = \int_5^\infty f_{X,\mu|\mu>5}(x,m)\, dm \neq f_X(x)$$